Published by Institute of Physics Publishing for SISSA/ISAS
Received: May 9, 2003
Accepted: June 14, 2003

Penrose limits of Lie Branes and a Nappi-Witten braneworld

Sonia Stanciu[✠]
Theoretical Physics Group, Imperial College
London, UK

José Figueroa-O'Farrill
Department of Mathematics, University of Edinburgh
Edinburgh, UK
E-mail: j.m.figueroa@ed.ac.uk

Abstract: Departing from the observation that the Penrose limit of $AdS_3 \times S^3$ is a group contraction in the sense of Inönü and Wigner, we explore the relation between the symmetric D-branes of $AdS_3 \times S^3$ and those of its Penrose limit, a six-dimensional symmetric plane wave analogous to the four-dimensional Nappi–Witten spacetime. Both backgrounds are Lie groups admitting bi-invariant lorentzian metrics and symmetric D-branes wrap their (twisted) conjugacy classes. We determine the (twisted and untwisted) symmetric D-branes in the plane wave background and we prove the existence of a space-filling D5-brane and, separately, of a foliation by D3-branes with the geometry of the Nappi-Witten spacetime which can be understood as the Penrose limit of the $AdS_2 \times S^2$ D3-brane in $AdS_3 \times S^3$. Parenthetically we also derive a simple criterion for a symmetric plane wave to be isometric to a lorentzian Lie group. In particular we observe that the maximally supersymmetric plane wave in IIB string theory is isometric to a lorentzian Lie group, whereas the one in M-theory is not.

Keywords: D-branes, Penrose limit and pp-wave background.

JHEP06(2003)025

© SISSA/ISAS 2003

Contents

1. Introduction

The Penrose limit [1, 2], originally a curiosity of four-dimensional General Relativity without many practical applications, has attracted much attention recently, largely due to its role in the novel large N limit of supersymmetric gauge theories discovered in [3] and vigorously studied since. The classical "pointlike" geometry of the Penrose limit is by now well understood [4], but strings probe other aspects of the spacetime geometry (e.g., D-submanifolds). It is therefore a natural question to ask how these behave under the Penrose limit.

This project thus originated as an attempt to understand what the Penrose limit, in its more recent stringy avatar, does to branes. In other words, if two string backgrounds are related by a Penrose limit, then how, if in any way, do the branes in one background relate to the branes in the other background? A priori one might suspect that there is little or no relation, since a brane might be localised far away from the null geodesic along which we are performing the Penrose limit, and hence will be lost in the limit. On the other hand, it is always possible to consider lorentzian branes which intersect a null geodesic and see if and how the limit of the background induces a limit of the brane.

JHEP06(2003)025

Lie groups admitting a bi-invariant metric comprise a class of exact string backgrounds whose D-submanifolds (at least the symmetric ones) are well understood geometrically. It is therefore reasonable to investigate these backgrounds first and study what happens to Lie branes[1] in the limit. In this paper we will collect our initial results in this topic by studying two vacua of the minimal chiral six-dimensional supergravity. These backgrounds can be lifted to exact backgrounds of ten-dimensional string theory by crossing them with some four-dimensional exact background like flat space or a K3 surface, for instance. However we will ignore the four-dimensional factor in what follows and concentrate on the six-dimensional geometry.

The backgrounds in question are $AdS_3 \times S^3$ and the six-dimensional version NW_6 of the Nappi-Witten spacetime [5] discovered in [6] in the supergravity context and denoted there KG_6. Both of these backgrounds are Lie groups and the Penrose limit which relates them can also be interpreted as a group contraction in the sense of Inönü–Wigner [7]. This will be explained in section 2, which also contains a subsection on the geometric characterisation of those symmetric plane waves which are isometric to Lie groups admitting a bi-invariant metric. Strings propagating in such Lie groups are described by a WZW model, whose symmetric D-submanifolds are known to be given by (shifted, twisted) conjugacy classes [8]–[11]. The symmetric D-submanifolds for $AdS_3 \times S^3$ were determined in [12, 13] and the results are reviewed in section 3 to ease the comparison. Those in NW_6 are worked out in section 4 for the first time following the method of [11] for the four-dimensional Nappi-Witten spacetime NW_4. In particular we demonstrate the existence of D3-branes isometric to NW_4 which foliate NW_6 in a variety of ways. This is done in two ways: by a geometric argument and by an explicit calculation. We also show that NW_6 admits a space-filling Lie brane. Both are examples of twisted conjugacy classes by a nontrivial outer automorphism.

Note added.

¿Y ha de morir contigo el mundo tuyo,
la vieja vida en orden tuyo y nuevo?
¿Los yunques y crisoles de tu alma
trabajan para el polvo y para el viento?

Antonio Machado

Most of the work contained in this paper was finished in March 2002, weeks before the terrible illness which so tragically cut short the life of the first named author began to take its toll. The decision to finally write up the paper at this late stage is due in part to the fact that some of the results presented here are revisited in forthcoming work involving the second named author, who would like to take this opportunity to apologise for the delay in completing the present paper.

In the intervening time a number of papers have appeared dealing with D-branes in the Penrose limit of $AdS_3 \times S^3$ in the supergravity approximation. Two papers [14, 15] deal with intersecting branes (at angles). As remarked at the end of [11, section 2.3], Lie branes

[1] Throughout this article we use the term “Lie brane” as a shorthand for “symmetric D-brane in a Lie group.”

JHEP06(2003)025

always wrap *submanifolds*, whence intersecting brane configurations are not described by twisted conjugacy classes, at least in a straightforward manner. Our results therefore bear no comparison. Three other papers can be compared and agree with our results. In [16] the authors construct D5- and NS5-brane solutions of type II supergravity whose worldvolume is the space-filling brane in NW_6 described in section 4.3.2 of the present paper. These solutions are obtained as the Penrose limits of fivebranes in $AdS_3 \times S^3 \times \mathbb{R}^4$ with worldvolumes which are space-filling in $AdS_3 \times S^3$. These fivebranes are not Lie branes, though. In [17], the authors construct an M5-brane solution whose worldvolume geometry agrees with the space-filling brane above by taking a Penrose limit of an M5-solution with worldvolume geometry $AdS_3 \times S^3$. Finally, in [18] the authors find, in the T-duality orbit of the D5-brane solution in [16], a D3-brane solution whose worldvolume is isometric to the Nappi-Witten spacetime NW_4 and hence to the symmetric D3-branes described in section 4.4. The Nappi-Witten spacetime also arises as a Penrose limit of the near horizon geometry of a NS5-brane [19].

2. The Penrose limits of $AdS_3 \times S^3$ as group contraction

The Penrose limits of $AdS_3 \times S^3$ were recently discussed in [20, 4]. There are two possible limits: the generic Penrose limit is a lorentzian symmetric space which, if the radii of curvature of AdS_3 and S^3 are the same, is conformally flat. There is a also a special limit in which the resulting spacetime is flat. In the generic case with equal radii of curvature, we can also understand this Penrose limit by isometrically embedding $AdS_3 \times S^3$ in $\mathbb{R}^8$ with a flat metric of signature $(6, 2)$. The Penrose limit is then induced by a generalised Penrose limit along a null plane through the origin, as discussed in [4].

2.1 Penrose limit as a group contraction

In this section we will discuss the Penrose limit in terms of group contraction. The manifold $AdS_3 \times S^3$ is locally isometric to the Lie group $\mathrm{SU}(1,1) \times \mathrm{SU}(2)$ with a bi-invariant metric. One-parameter subgroups are geodesic relative to bi-invariant metrics, hence any one-parameter subgroup which is null relative to the bi-invariant metric gives rise to a null geodesic whose associated Penrose limit can be understood as a group contraction. Such special cases of Penrose limits have been considered in [21].

We will identify the Lie groups $\mathrm{SU}(1,1)$ and $\mathrm{SU}(2)$ with their defining representations. That means that $\mathrm{SU}(1,1)$ is the group of matrices of the form

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \tag{2.1}$$

where a and b are complex numbers satisfying $|a|^2 - |b|^2 = 1$. Similarly $\mathrm{SU}(2)$ is the group of matrices of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \tag{2.2}$$

JHEP06(2003)025

where a and b are complex numbers satisfying $|a|^2+|b|^2=1$. The groups $\mathrm{SU}(1,1)$ and $\mathrm{SU}(2)$ admit bi-invariant metrics induced from ad-invariant inner products in their respective Lie algebras:

$$\langle X, Y\rangle = \pm\frac{1}{2}\operatorname{Tr} XY\,, \tag{2.3}$$

where the positive sign is for $X, Y \in \mathfrak{su}(1,1)$ and the negative sign for $X, Y \in \mathfrak{su}(2)$.

Consider the $\mathrm{U}(1)$ subgroup of $\mathrm{SU}(1,1)\times\mathrm{SU}(2)$ consisting of diagonal matrices

$$\begin{pmatrix} e^{i\varphi} & & & \\ & e^{-i\varphi} & & \\ & & e^{i\varphi} & \\ & & & e^{-i\varphi}\end{pmatrix} \qquad \text{where } \varphi \in \mathbb{R}/2\pi\mathbb{Z}. \tag{2.4}$$

This subgroup defines a null circle in $\mathrm{SU}(1,1)\times\mathrm{SU}(2)$ relative to the bi-invariant metric. Indeed, the tangent vector to the circle at the above element is

$$\begin{pmatrix} ie^{i\varphi} & & & \\ & -ie^{-i\varphi} & & \\ & & ie^{i\varphi} & \\ & & & -ie^{-i\varphi}\end{pmatrix},$$

whose norm vanishes relative to the inner product defined by (2.3).

Let X_i, $i=0,1,\ldots,5$ be the following pseudo-orthonormal basis for $\mathfrak{su}(1,1)\oplus\mathfrak{su}(2)$:

$$\begin{aligned} \mathsf{X}_0 &= i\sigma_3\oplus 0 & \mathsf{X}_1 &= \sigma_1\oplus 0 & \mathsf{X}_2 &= \sigma_2\oplus 0\\ \mathsf{X}_3 &= 0\oplus i\sigma_1 & \mathsf{X}_4 &= 0\oplus i\sigma_2 & \mathsf{X}_5 &= 0\oplus i\sigma_3\,, \end{aligned}$$

where the σ_i are the standard (hermitean) Pauli matrices. The nonzero Lie brackets are given by

$$\begin{aligned} [\mathsf{X}_0,\mathsf{X}_1] &= -2\mathsf{X}_2 & [\mathsf{X}_5,\mathsf{X}_3] &= -2\mathsf{X}_4\\ [\mathsf{X}_0,\mathsf{X}_2] &= 2\mathsf{X}_1 & [\mathsf{X}_5,\mathsf{X}_4] &= 2\mathsf{X}_3\\ [\mathsf{X}_1,\mathsf{X}_2] &= 2\mathsf{X}_0 & [\mathsf{X}_3,\mathsf{X}_4] &= -2\mathsf{X}_5\,, \end{aligned}$$

and the inner product is

$$\langle \mathsf{X}_i, \mathsf{X}_j\rangle = \eta_{ij}\,,$$

where $\eta = \operatorname{diag}(-1,1,\ldots,1)$.

The Lie algebra of the circle subgroup is generated by $\mathsf{U}=\mathsf{X}_0+\mathsf{X}_5$. We see from the above inner product that U is indeed null. Let $\mathsf{V}=\mathsf{X}_0-\mathsf{X}_5$ be the complementary null generator. In order to define the contraction, we let $\Omega>0$ and introduce new generators by

$$\mathsf{P}_i = \Omega\mathsf{X}_i \qquad \mathsf{J}=\frac{1}{2}\mathsf{U} \qquad \mathsf{K}=\Omega^2\mathsf{V}\,,$$

for $i=1,2,3,4$. The contracted Lie algebra $\mathfrak{n}$ is defined as the limit $\Omega\to 0$ of the brackets of the new generators:

$$\begin{aligned} [\mathsf{J},\mathsf{P}_1] &= -\mathsf{P}_2 & [\mathsf{J},\mathsf{P}_3] &= -\mathsf{P}_4 & [\mathsf{P}_1,\mathsf{P}_2] &= \mathsf{K}\\ [\mathsf{J},\mathsf{P}_2] &= \mathsf{P}_1 & [\mathsf{J},\mathsf{P}_4] &= \mathsf{P}_3 & [\mathsf{P}_3,\mathsf{P}_4] &= \mathsf{K} \end{aligned}$$

JHEP06(2003)025

with K central. The resulting Lie algebra is solvable: indeed its second derived ideal is central. It is essentially a Heisenberg Lie algebra with central element K, together with an outer automorphism J. It admits an ad-invariant inner product inherited from the one in $\mathfrak{su}(1,1) \oplus \mathfrak{su}(2)$. In fact, the inner product on the generators $\{\mathsf{P}_i, \mathsf{J}, \mathsf{K}\}$ is given by taking the limit $\Omega \to 0$ of $\Omega^{-2}\langle -, - \rangle$, where $\langle -, - \rangle$ is the inner product in (2.3). The nonzero inner products are

$$\langle \mathsf{P}_i, \mathsf{P}_j \rangle = \delta_{ij} \qquad \langle \mathsf{J}, \mathsf{K} \rangle = -1\,,$$

where $i, j = 1, 2, 3, 4$.

We can view this Lie algebra $\mathfrak{n}$ from another point of view, which is more useful in determining its conjugacy classes. We can understand it is a central extension of a subalgebra of the four-dimensional euclidean algebra. Indeed, the generators P_i are translations in $\mathbb{R}^4$ and J is a combined rotation in the (12) and (34) planes. Together they span a subalgebra of the euclidean algebra $\mathfrak{so}(4) \ltimes \mathbb{R}^4$. Introducing the complex structure

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

we can rewrite the Lie algebra as

$$[\mathsf{J}, \mathsf{P}_i] = \sum_j J_{ij} \mathsf{P}_j \qquad [\mathsf{P}_i, \mathsf{P}_j] = -J_{ij}\mathsf{K}\,.$$

In the language of [22] (see also [23]) this exhibits the Lie algebra as the *double extension* of the abelian Lie algebra $\mathbb{R}^4$ by $\mathbb{R}$, which explains why this is a solvable Lie group admitting a bi-invariant inner product.

2.2 A six-dimensional Nappi-Witten group

The interpretation of the Lie algebra $\mathfrak{n}$ as a central extension of a subalgebra of the euclidean algebra makes it very easy to write down the corresponding Lie group $\mathcal{N}$. It will prove convenient to think of $\mathbb{R}^4$ with the complex structure J as $\mathbb{C}^2$. Indeed, on the complex linear combinations $\mathsf{P}_1 + i\mathsf{P}_2$ and $\mathsf{P}_3 + i\mathsf{P}_4$, the generator J acts by multiplication by i. Let us define $R(\theta) = \exp(\theta\mathsf{J})$ and $T(u) = \exp(u \cdot \mathsf{P})$, where for $u \in \mathbb{C}^2$, $u \cdot \mathsf{P} = \operatorname{Re} \bar{u}^t \mathsf{P}$, where $\mathsf{P} = (\mathsf{P}_1 + i\mathsf{P}_2, \mathsf{P}_3 + i\mathsf{P}_4)$. We then have the following group multiplications

$$\begin{aligned} R(\theta)R(\theta') &= R(\theta + \theta') \\ R(\theta)T(u) &= T\left(e^{-i\theta}u\right) R(\theta)\,. \end{aligned} \tag{2.5}$$

The central extension now makes the translation algebra noncommutative. Indeed, using the Baker-Campbell-Hausdorff formula it is easy to find

$$T(u)T(u') = T(u + u')Z\left(\frac{1}{2}\omega(u, u')\right), \tag{2.6}$$

JHEP06(2003)025

where $Z(t) = \exp(t\mathsf{K})$ and where ω is the symplectic structure defined by

$$\omega\left(u, u'\right) = \mathrm{Im}\left(\bar{u}^t u'\right).$$

Therefore the group elements of (the universal covering group of) $\mathcal{N}$ are parametrised by $(u, \theta, t) \in \mathbb{C}^2 \times \mathbb{R} \times \mathbb{R}$ and the corresponding group element $g(u, \theta, t)$ is given by

$$g(u, \theta, t) = T(u)R(\theta)Z(t).$$

It is easy to work out the group multiplication law from equations (2.5) and (2.6), and one finds:

$$g(u, \theta, t)g(u', \theta', t') = g\left(u + e^{-i\theta}u', \theta + \theta', t + t' + \frac{1}{2}\omega(u, e^{-i\theta}u')\right). \tag{2.7}$$

It follows that the identity is $g(0, 0, 0)$ and that the inverse of $g(u, \theta, t)$ is given by

$$g(u, \theta, t)^{-1} = g\left(-e^{i\theta}u, -\theta, -t\right). \tag{2.8}$$

Finally let us work out the bi-invariant metric on the group in this (global) coordinate system. The metric at (u, θ, t) is given by

$$ds^2 = \langle g^{-1}dg, g^{-1}dg\rangle, \tag{2.9}$$

where $g = g(u, \theta, t)$. A simple calculation shows that

$$g^{-1}dg(u, \theta, t) = \left(e^{i\theta}du\right) \cdot \mathsf{P} + d\theta\mathsf{J} + \left(dt - \frac{1}{2}\omega(u, du)\right)\mathsf{K}. \tag{2.10}$$

Explicitly, if $u = (u_1 + iu_2, u_3 + iu_4)$, one has

$$\begin{aligned}\left(e^{i\theta}du\right) \cdot \mathsf{P} &= (du_1 \cos\theta - du_2 \sin\theta)\mathsf{P}_1 + (du_2 \cos\theta + du_1 \sin\theta)\mathsf{P}_2 + \\ &\quad + (du_3 \cos\theta - du_4 \sin\theta)\mathsf{P}_3 + (du_4 \cos\theta + du_3 \sin\theta)\mathsf{P}_4.\end{aligned}$$

Inserting this into (2.9), we obtain the following expression for the bi-invariant metric on the group:

$$ds^2 = |du|^2 + \omega(u, du)d\theta - 2d\theta dt.$$

Changing coordinates to $x = e^{i\theta/2}u$, $x^+ = -2t$, $x^- = \frac{1}{2}\theta$, the metric becomes

$$ds^2 = |dx|^2 - |x|^2(dx^-)^2 + 2dx^+dx^-, \tag{2.11}$$

which we recognise as a Cahen–Wallach metric, corresponding to a (conformally flat) indecomposable lorentzian symmetric space [24]. In fact, it is a six-dimensional analogue of the Nappi-Witten spacetime in [5] and hence will be denoted NW_6 when we want to emphasise its geometry instead of the Lie group structure. As a vacuum solution to the minimal chiral six-dimensional supergravity, it was discovered by Meessen [6].

The fact that a symmetric space is isometric to a Lie group with a bi-invariant metric is of course not unusual. After all, every group G admitting a bi-invariant metric is itself a symmetric space: simply consider $(G \times G)/G$, where we quotient by the diagonal subgroup.

JHEP06(2003)025

However for the Cahen-Wallach symmetric spaces of the type considered here, the usual description is G/K where G is a solvable Lie group and K an abelian subgroup. The question arises as to whether every Cahen-Wallach space is isometric to some Lie group with a bi-invariant metric. Before turning our attention to the description of symmetric branes for this geometry, let us take a moment to answer this question, since the answer might be of independent interest.

2.3 Symmetric plane waves and lorentzian Lie groups

It follows from the structure theorem of [22] and the refinement in [23, 25] that indecomposable lorentzian Lie groups are either simple (e.g., $\mathrm{SU}(1,1)$) or solvable. In order to obtain a symmetric plane wave metric we need only concentrate on the solvable case. The only solvable Lie groups admitting a bi-invariant metric are those whose Lie algebras are obtained from the one-dimensional Lie algebra by iterating two constructions: double extension and orthogonal direct sum. Furthermore if the metric is lorentzian then there is at most one double extension. Let us focus on indecomposable Lie algebras admitting a lorentzian invariant metric. In this case there is precisely one double extension and in fact it is not hard to show that these Lie algebras are given by the double extension of an abelian Lie algebra $\mathbb{E}^n$ (with the standard euclidean inner product $\langle -,-\rangle$) by the one-dimensional Lie algebra $\mathbb{R}$. The action of $\mathbb{R}$ on $\mathbb{E}^n$ is generated by a skew-symmetric endomorphism $J : \mathbb{E}^n \to \mathbb{E}^n$. If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}^n$ and e_- generates $\mathbb{R}$ and e_+ generates the dual $\mathbb{R}^*$, then the Lie brackets are given by

$$[e_-, \boldsymbol{x}] = J(\boldsymbol{x}) \qquad \text{and} \qquad [\boldsymbol{x}, \boldsymbol{y}] = \omega(\boldsymbol{x}, \boldsymbol{y}) e_+ \,,$$

where $\omega(\boldsymbol{x}, \boldsymbol{y}) = \langle J(\boldsymbol{x}), \boldsymbol{y}\rangle$. This is a solvable Lie algebra admitting an invariant scalar product extending $\langle -,-\rangle$ on $\mathbb{E}^n$ by declaring $\langle e_-, e_+\rangle = 1$. The algebra will be indecomposable if J is non-degenerate, which requires n to be even. Let G_ω denote the 1-connected solvable Lie group with this Lie algebra. The invariant scalar product on the Lie algebra induces a bi-invariant metric on G_ω, which relative to coordinates similar to the ones used above, takes the Cahen-Wallach form

$$ds^2 = 2dx^+ dx^- - \langle J\boldsymbol{x}, J\boldsymbol{x}\rangle \, (dx^-)^2 + \langle d\boldsymbol{x}, d\boldsymbol{x}\rangle \,.$$

Recall that a general Cahen-Wallach metric depends on a symmetric matrix A in the form

$$ds^2 = 2dx^+ dx^- + A(\boldsymbol{x}, \boldsymbol{x})(dx^-)^2 + \langle d\boldsymbol{x}, d\boldsymbol{x}\rangle \,.$$

Comparing the two metrics we see that a Cahen-Wallach metric is the bi-invariant metric on a solvable Lie group if and only if the matrix $A = J^2$ is negative-definite and every eigenvalue has even multiplicity. In particular this means that the metric of the IIB maximally supersymmetric wave [26] is in fact a bi-invariant metric on a solvable Lie group, whereas the metric of the maximally supersymmetric M-wave [27] is not. Of course, strings propagating in the IIB wave do not really see the Lie group structure[2] since this background has no B-field, but rather the RR self-dual five-form.

[2] although it might see a 4-Lie group structure [28], if such a thing exists.

JHEP06(2003)025

3. Lie branes in $\mathrm{SU}(1,1) \times \mathrm{SU}(2)$

The Lie branes in $\mathrm{SU}(1,1)\times\mathrm{SU}(2)$ wrap submanifolds which are given by (twisted, shifted) conjugacy classes of the group. These were analysed originally in [12] and [13]. We briefly review these results to ease the comparison.

The conjugacy classes of $\mathrm{SU}(2)$ are parametrised by $T/\mathbb{Z}_2$, where T is a maximal torus and $\mathbb{Z}_2$ is the Weyl group which acts with two fixed points. The quotient is therefore an interval, which we can take to be $[0,\pi]$. The conjugacy class corresponding to $\theta \in [0,\pi]$ is a round 2-sphere with radius $\sin^2\theta$, hence it degenerates to a point at the endpoints of the interval. The induced metric on the two-dimensional spheres is euclidean. The group $\mathrm{SU}(2)$ does not have outer automorphisms, hence no twisted D-branes.

If we parametrise $\mathrm{SU}(1,1)$ as follows

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} x+iy & u+iv \\ u-iv & x-iy \end{pmatrix} \middle| \, x^2+y^2 = 1+u^2+v^2 \right\},$$

then conjugacy classes are essentially the intersection of the hyperboloid $x^2+y^2 = 1+u^2+v^2$ in $\mathbb{E}^{2,2}$ with the affine hyperplanes $x =$ constant. More precisely, when $|x| = 1$ each of the resulting intersections breaks up into three conjugacy classes corresponding to the apex of a cone and the upper and lower deleted cones. Similarly when $|x| < 1$ the intersection is a two-sheeted hyperboloid and each sheet is a conjugacy class. Keeping only those conjugacy classes on which the metric is nondegenerate, we have (see [12] for more details):

1. 2 pointwise classes corresponding to $\pm\mathbb{1}$;
2. a family of two-dimensional lorentzian submanifolds isometric to dS_2 with (squared) radius of curvature proportional to x^2-1; and
3. a family of two-dimensional non-flat riemannian submanifolds isometric to hyperbolic spaces with (squared) radius of curvature proportional to $1-x^2$.

The group $\mathrm{SU}(1,1)$ does have outer automorphisms, giving rise to twisted conjugacy classes. Up to inner automorphisms there is a unique nontrivial outer automorphism which is realised by complex conjugation in the fundamental representation. The twisted conjugacy classes are one-sheeted hyperboloids which can be understood as the intersection of the hyperboloid $x^2+y^2 = 1+u^2+v^2$ with the affine hyperplanes $u =$ constant. The corresponding metric is lorentzian and has negative constant sectional curvature, whence it is isometric to an embedded AdS_2 in AdS_3. These twisted D-branes were discovered by Bachas and Petropoulos [13].

We can now enumerate the possible Lie branes in $AdS_3 \times S^3$:

1. pointlike D-instantons;
2. D-strings sitting at a point in S^3, with worldvolume geometries $\mathrm{AdS}_2 \subset \mathrm{AdS}_3$ or $\mathrm{dS}_2 \subset \mathrm{AdS}_3$;
3. hyperbolic euclidean D-strings sitting at a point in S^3;

JHEP06(2003)025

4. spherical euclidean D-strings sitting at a point in AdS_3;

5. D3-branes isometric to $AdS_2 \times S^2$ or $\mathrm{dS}_2 \times S^2$; and

6. euclidean D3-branes isometric to $H_2 \times S^2$.

This is now to be compared with the Lie branes in the contracted group, to which we now turn.

4. Lie branes in the six-dimensional Nappi-Witten group

The problem of determining the submanifolds wrapped by Lie branes in the six-dimensional Nappi-Witten group $\mathcal{N}$ obtained by contracting $\mathrm{SU}(1,1) \times \mathrm{SU}(2)$ consists in determining its (twisted) conjugacy classes. As we will see, this problem is very similar to that for the four-dimensional Nappi-Witten group [5], which was solved in [11], albeit computationally somewhat more involved.

4.1 Orthogonal automorphisms

The first step is to determine the group of orthogonal automorphisms of the Nappi-Witten group $\mathcal{N}$. Since the exponential map is a diffeomorphism in this case, we need only determine the orthogonal automorphisms of its Lie algebra $\mathfrak{n}$. These are Lie algebra automorphisms which preserve the metric. It is straightforward to show that the most general orthogonal automorphism $\tau : \mathfrak{n} \to \mathfrak{n}$ is given by

$$\begin{aligned}
\tau(\mathsf{P}_i) &= \sum_j M_{ji} \mathsf{P}_j + \varepsilon \sum_j M_{ji} L_j \mathsf{K} \\
\tau(\mathsf{J}) &= \sum_i L_i \mathsf{P}_i + \varepsilon \mathsf{J} + \frac{1}{2} \varepsilon \sum_i L_i L_i \mathsf{K} \\
\tau(\mathsf{K}) &= \varepsilon \mathsf{K} ,
\end{aligned} \tag{4.1}$$

where L_i are arbitrary, $\varepsilon = \pm 1$ and M_{ij} satisfy

$$\sum_k M_{ki} M_{kj} = \delta_{ij} \qquad \text{and} \qquad \sum_{k,\ell} M_{ki} J_{k\ell} M_{\ell j} = \varepsilon J_{ij} .$$

In other words, M_{ij} are the entries of an orthogonal matrix $M \in \mathrm{O}(4)$ which preserves the complex structure J up to a sign. If $\varepsilon = 1$, then M preserves the complex structure and hence the induced orientation in $\mathbb{R}^4$. This means that M belongs to the intersection of $\mathrm{SO}(4)$ and $\mathrm{GL}(2,\mathbb{C})$ in $\mathrm{GL}(4,\mathbb{R})$, which is isomorphic to $\mathrm{U}(2)$. If $\varepsilon = -1$, then $M = UX$ where U belongs to $\mathrm{U}(2)$ and X is any matrix in $\mathrm{SO}(4)$ which obeys $XJ = -JX$. One such possibility is $X = \mathrm{diag}(1,-1,1,-1)$, which belongs to the normaliser of $\mathrm{U}(2)$ in $\mathrm{SO}(4)$. In other words, conjugation by X defines an outer automorphism χ of $\mathrm{U}(2) \subset \mathrm{SO}(4)$ which in fact is none other than complex conjugation. We therefore have two types of elements $(L, U, 1)$ and $(L, UX, -1)$, where $U \in \mathrm{U}(2) \subset \mathrm{SO}(4)$ and $L \in \mathbb{C}^2 \cong \mathbb{R}^4$. It is easy to work out the multiplication law for this group:

$$(L, M, \varepsilon) \cdot \left(L', M', \varepsilon'\right) = \left(ML' + \varepsilon' L, MM', \varepsilon\varepsilon'\right) .$$

JHEP06(2003)025

This shows that the group $\mathrm{Aut}_o(\mathfrak{n})$ of orthogonal automorphisms of $\mathfrak{n}$ (and hence of $\mathcal{N}$) is isomorphic to

$$\mathrm{Aut}_o(\mathfrak{n}) \cong \mathbb{C}^2 \rtimes (\mathrm{U}(2) \rtimes \mathbb{Z}_2) \ ,$$

where $\mathrm{U}(2) \rtimes \mathbb{Z}_2$, the principal extension of $\mathrm{U}(2)$ by the outer automorphism χ, acts on $\mathbb{C}^2$ as follows. We embed $\mathbb{C}^2 \subset \mathrm{Aut}_o(\mathfrak{n})$ as $L \mapsto (L, \mathbb{1}, 1)$ and the action of $\mathrm{U}(2) \rtimes \mathbb{Z}_2$ is induced by conjugation in $\mathrm{Aut}_o(\mathfrak{n})$, whence

$$U \cdot L = UL \qquad \text{and} \qquad UX \cdot L = -U\bar{L} \, .$$

Notice that the invariant subgroup consisting of inner automorphisms is $\mathbb{C}^2 \rtimes \mathrm{U}(1)$, where $\mathrm{U}(1) \subset \mathrm{SO}(4)$ is the circle subgroup generated by J. Let us define the group of outer (orthogonal) automorphisms:

$$\mathrm{Out}_o(\mathfrak{n}) = \frac{\mathrm{Aut}_o(\mathfrak{n})}{\mathrm{Inn}_o(\mathfrak{n})} \cong \frac{\mathbb{C}^2 \rtimes (\mathrm{U}(2) \rtimes \mathbb{Z}_2)}{\mathbb{C}^2 \rtimes \mathrm{U}(1)} \, .$$

Notice that every element $(L, M, \varepsilon) \in \mathrm{Aut}_o(\mathfrak{n})$ can be decomposed uniquely as

$$(L, M, \varepsilon) = (0, M, \varepsilon)(M^{-1}L, \mathbb{1}, 1) \, ,$$

whence modulo the inner automorphisms we can put $L = 0$. In other words, the group of outer automorphisms is isomorphic to

$$\mathrm{Out}_o(\mathfrak{n}) \cong \frac{\mathrm{U}(2) \rtimes \mathbb{Z}_2}{\mathrm{U}(1)} \, .$$

Further, observe that a given element $(M, \varepsilon) \in \mathrm{U}(2) \rtimes \mathbb{Z}_2$, is such that $M = U$ if $\varepsilon = 1$ or $M = UX$ if $\varepsilon = -1$, where $U \in \mathrm{U}(2)$. Given $U \in \mathrm{U}(2)$, we can write it as $U = S\delta^{1/2}$, where $S \in \mathrm{SU}(2)$ and $\delta = \det \mathrm{U}$. This implies that we can write

$$\begin{aligned} (U, 1) &= (S, 1) \left(\delta^{1/2}, 1\right) \\ (UX, -1) &= (SX, -1) \left(\bar{\delta}^{1/2}, 1\right) , \end{aligned}$$

whence modulo inner automorphisms we can always choose U to lie in $\mathrm{SU}(2)$. Moreover U and $-U$ are equivalent modulo inner automorphisms, whence we have that

$$\mathrm{Out}_o(\mathfrak{n}) \cong \frac{\mathrm{SU}(2) \rtimes \mathbb{Z}_2}{\mathbb{Z}_2} \, .$$

In other words, $\mathrm{Out}_o(\mathfrak{n})$ consists of elements of the form $(S, 1)$ and $(SX, -1)$, but where S and $-S$ give the same outer automorphism. This result is to be contrasted with the four-dimensional Nappi-Witten group, whose group of outer (orthogonal) automorphisms is finite of order 2.

JHEP06(2003)025

4.2 Untwisted Lie branes

The untwisted Lie branes wrap conjugacy classes of the group $\mathcal{N}$. To determine these classes we first compute the adjoint action of $\mathcal{N}$ on $\mathcal{N}$. To determine their geometry we use the fact that at a point $g_0 \in \mathcal{N}$, the tangent space $T_{g_0}\mathcal{N}$ has two natural subspaces: $T_{g_0}\mathcal{C}$, which is the tangent space to the conjugacy class $\mathcal{C}$ and $T_{g_0}\mathcal{Z}$, the tangent space to the centraliser subgroup $\mathcal{Z}$ of g_0: notice that trivially $g_0 \in \mathcal{Z}$. It can be shown that $T_{g_0}\mathcal{Z} = (T_{g_0}\mathcal{C})^\perp$, but it may happen that $T_{g_0}\mathcal{Z} \cap T_{g_0}\mathcal{C} \neq \{0\}$. This happens if and only if the restriction of the metric to the conjugacy class $\mathcal{C}$ (or equivalently to the centraliser $\mathcal{Z}$) is degenerate. In this case, we cannot interpret the conjugacy class as a D-submanifold, at least in a straightforward manner, and we will ignore these cases presently. If the metric restricts to $\mathcal{C}$ non-degenerately, then $T_{g_0}\mathcal{N} = T_{g_0}\mathcal{C} \oplus T_{g_0}\mathcal{Z}$ and the direct sum is orthogonal. We can then easily determine the signature of the metric on the conjugacy class by determining that of the metric on the centraliser subgroup, which is often easier to determine, since bi-invariance allows us to work at the identity and hence with the Lie algebra.

From equations (2.7) and (2.8) we see that

$$\begin{aligned} &g(u,\theta,t)g(u_0,\theta_0,t_0)g(u,\theta,t)^{-1} = \\ &\quad = g\left(\left(1-e^{-i\theta_0}\right)u + e^{-i\theta}u_0, \theta_0, t_0 - \frac{1}{2}\operatorname{Im}\left(\bar{u}_0^t e^{i\theta}\left(1+e^{-i\theta_0}\right)\right)\right), \end{aligned} \tag{4.2}$$

from where we deduce that θ_0 is an invariant of the conjugacy class; whence the induced metric is euclidean, although possibly degenerate. Hence we fix a value of θ_0 and investigate the action on the remaining coordinates (u_0, t_0). Furthermore, periodicity allows us to restrict our attention to $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$.

$\theta_0 = 0 \pmod{2\pi}$. Conjugation maps

$$(u_0, t_0) \mapsto \left(e^{-i\theta}u_0, t_0 - \operatorname{Im}\left(\bar{u}_0^t e^{i\theta} u\right)\right).$$

If $u_0 = 0$, then $g(0,0,t_0)$ belongs to the centre and hence the conjugacy class is a point. If $u_0 \neq 0$ we obtain the cylindrical conjugacy class consisting of elements

$$g(e^{-i\theta}u_0, 0, t) \qquad \text{for all } t \in \mathbb{R} \text{ and } \theta \in \mathbb{R}/2\pi\mathbb{Z},$$

which is diffeomorphic to $\mathbb{R} \times S^1$. A routine calculation shows that the metric restricts degenerately, whence these cylindrical conjugacy classes cannot be straightforwardly interpreted as D-branes.

$\theta_0 = \pi \pmod{2\pi}$. Conjugation maps

$$(u_0, t_0) \mapsto \left(2u + e^{-i\theta}u_0, t_0\right),$$

which is the 4-plane labelled by t_0. It is easy to see that the induced metric has euclidean signature. Thus these conjugacy classes are wrapped by euclidean D3-branes.

JHEP06(2003)025

$\theta_0 \neq 0, \pi \pmod{2\pi}$. We first determine the centraliser of an element $g(u_0, \theta_0, t_0)$. We find that it consists of the elements

$$g(u(\theta), \theta, t) \quad \text{where} \quad u(\theta) = \frac{1 - e^{-i\theta}}{1 - e^{-i\theta_0}} u_0 \, .$$

Since the centraliser is two-dimensional (parametrised by θ and t), the conjugacy classes are four-dimensional. The metric on the centraliser is lorentzian, whence the metric on the conjugacy class at the chosen element (and by homogeneity everywhere) is euclidean, whence these conjugacy classes are wrapped by euclidean D3-branes.

In summary, the conjugacy classes can be wrapped by D-instantons and euclidean D3-branes.

4.3 Twisted Lie branes

We now turn our attention to the twisted conjugacy classes

$$C^r(g_0) = \left\{ r(g) g_0 g^{-1} | g \in \mathcal{N} \right\} ,$$

where r is an orthogonal automorphism. As discussed above, the orthogonal outer automorphism group consists of elements of the form (M, ε), where $M = S$ if $\varepsilon = 1$ and $M = SX$ if $\varepsilon = -1$, where $S \in \mathrm{SU}(2)$. We repeat that S and $-S$ give rise to the same outer automorphism. If $g = g(z, \theta, t)$ is a group element and r is the automorphism corresponding to (M, ε), then

$$r(g) = g(Mz, \varepsilon\theta, \varepsilon t) = \begin{cases} g(Sz, \theta, t) & \text{if } \varepsilon = 1 \\ g(S\bar{z}, -\theta, -t) & \text{if } \varepsilon = -1. \end{cases}$$

Under twisted conjugation by g, the element $g_0 = g(z_0, \theta_0, t_0)$ is mapped to

$$r(g) g_0 g^{-1} = g(z', \theta', t') \, ,$$

where, if $\varepsilon = 1$,

$$\begin{aligned} z' &= Sz + e^{-i\theta} z_0 - e^{-i\theta_0} z \\ \theta' &= \theta_0 \\ t' &= t_0 + \frac{1}{2}\omega\left(Sz, e^{-i\theta} z_0\right) - \frac{1}{2}\omega\left(Sz + e^{-i\theta} z_0, e^{-i\theta_0} z\right) ; \end{aligned} \tag{4.3}$$

whereas, if $\varepsilon = -1$,

$$\begin{aligned} z' &= S\bar{z} + e^{i\theta} z_0 - e^{-i(\theta_0 - 2\theta)} z \\ \theta' &= \theta_0 - 2\theta \\ t' &= t_0 - 2t + \frac{1}{2}\omega\left(S\bar{z}, e^{i\theta} z_0\right) - \frac{1}{2}\omega\left(S\bar{z} + e^{i\theta} z_0, e^{-i(\theta_0 - 2\theta)} z\right) . \end{aligned} \tag{4.4}$$

JHEP06(2003)025

4.3.1 The case $\varepsilon = 1$

Let us first consider the case of $\varepsilon = 1$. We notice that θ_0 is an orbit invariant, whence the metric on the orbit will be euclidean, but perhaps degenerate. The stabiliser of g_0 is the subgroup consisting of elements of the form $g(z, \theta, t)$ where t is unconstrained, and where z, θ satisfy

$$\begin{aligned} \left(S - e^{-i\theta_0}\right) z &= \left(1 - e^{-i\theta}\right) z_0 \\ \omega\left(Sz, e^{-i\theta} z_0\right) &= \omega\left(Sz + e^{-i\theta} z_0, e^{-i\theta_0} z\right). \end{aligned} \tag{4.5}$$

Inserting the first equation into the second, allows us to rewrite it as

$$\omega\left(e^{-i\theta_0} z + z_0, \left(1 + e^{-i\theta}\right) z_0\right) = 0\,. \tag{4.6}$$

Two cases present themselves according to whether z_0 vanishes or not.

If $z_0 = 0$ then the second equation is automatically satisfied, θ is unconstrained and z satisfies

$$\left(S - e^{-i\theta_0}\right) z = 0\,.$$

Therefore the resulting stabiliser consists of elements $g(z, \theta, t)$ where z obeys the above equation. We can break this up further into three cases depending on the rank of the complex linear map $\varphi := S - e^{-i\theta_0}$. In the generic case, φ will have (complex) rank 2 and hence $z = 0$. The resulting stabiliser consists of $g(0, \theta, t)$ which is two-dimensional, whence the orbit is four-dimensional: a euclidean D3-brane. If φ has rank 1, so that $e^{i2\theta_0} \neq 1$, then the stabiliser is four-dimensional, and hence the orbit is a euclidean D-string. Finally the case where φ has zero rank, which can only happen when $e^{i2\theta_0} = 1$ and $S = \pm\mathbb{1}$, reduces to an inner automorphism. This has been considered in the previous section and one obtains a D-instanton.

Now suppose that $z_0 \neq 0$. We again distinguish several cases depending on the rank of φ, although now z is not necessarily in its kernel. In the generic case of rank 2, then φ is invertible and we can solve for z:

$$z = \varphi^{-1}\left(1 - e^{-i\theta}\right) z_0\,.$$

Inserting this into the second equation one finds that (remarkably?) the equation is automatically satisfied for any value of θ. Therefore the stabiliser, which consists of elements $g(z(\theta), \theta, t)$, is clearly two-dimensional and hence, for generic θ_0, the twisted conjugacy classes are wrapped by euclidean D3-branes.

There are two types of non-generic θ_0: depending on whether the (complex) rank of φ is 1 or 0. This latter case can only happen when $S = \pm\mathbb{1}$ and $\theta_0 = k\pi$ for $k \in \mathbb{Z}$. This latter case corresponds to twisting by an inner automorphism, whence it was discussed in the previous section. We will therefore focus briefly on the case of φ having rank 1, and $S \neq \pm\mathbb{1}$.

There are two cases that we must consider, depending on whether or not z_0 belongs to the image of φ. If $z_0 \notin \operatorname{im}\varphi$ then the first equation in (4.5) can only be satisfied if

JHEP06(2003)025

$e^{i\theta} = 1$ and $z \in \ker\varphi$. The second equation $\omega(z, e^{i\theta_0}z_0) = 0$ either gives none or one (real) equation on z, so that the stabiliser subgroup is either three-dimensional or two-dimensional respectively. The resulting Lie branes are then euclidean D2- or D3-branes, provided the metric restricts non-degenerately. We have not checked, since we do not think they are particularly interesting.

Finally let us consider the case where $z_0 = \varphi(w_0)$ for some w_0. In this case $z = (1 - e^{-i\theta})w_0 + w$, where $w \in \ker\varphi$. Since $z_0 \neq 0$, φ has rank 1 or 2. The case of full rank has $z = z(\theta)$ and then the second equation in (4.5) fixes θ. As a result the stabiliser is one-dimensional, whence the orbit if five-dimensional but inherits a degenerate metric. If the rank of φ is not maximal, then the stabiliser has dimension three and, provided the metric is not degenerate (which we have not checked) the orbit can be wrapped by euclidean D2-branes.

In summary, all the Lie branes wrapping twisted conjugacy classes associated to outer automorphisms with $\varepsilon = 1$ are euclidean: D-instantons, D-strings, D2-branes (possibly) and D3-branes.

4.3.2 The case $\varepsilon = -1$

From the explicit expression (4.4) for the twisted conjugacy of an element $g(z_0, \theta_0, t_0)$, we find that the stabiliser subgroup consists of elements $g(z, 0, t)$ where[3]

$$4t = \omega(S\bar{z}, z_0) - \omega\left(S\bar{z} + z_0, e^{-i\theta_0}z\right),$$

and z satisfies the *real* linear equation

$$S\bar{z} = e^{-i\theta_0}z\,.$$

Inserting this equation into the above expression for t we find

$$t = \frac{1}{2}\omega\left(e^{-i\theta_0}z, z_0\right).$$

The dimension of the orbit depends the rank of the real linear map $\varrho : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $z \mapsto S\bar{z} - e^{-i\theta_0}z$, where $S \in \mathrm{SU}(2)$. A straightforward calculation shows that ϱ is invertible *unless* the off-diagonal entries of S are pure imaginary. Concretely, if $S = \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a}\end{smallmatrix}\right)$, then $\det\varrho = 4(\operatorname{Im} b)^2$. Moreover if $\operatorname{Im} b = 0$ then the (real) rank of ϱ is 2. Therefore, for generic S, the stabiliser is trivial and the resulting Lie brane is a space-filling D5-brane. If S is such that ϱ has rank 2 then the resulting Lie brane is a D3-brane. We will see below that these D3-branes have the geometry of a four-dimensional Nappi-Witten spacetime. We see thus that the six-dimensional Nappi-Witten group admits many foliations by four-dimensional Nappi-Witten spacetimes.

To summarise the results of this section, the Lie branes of NW_6 are of the following types: D-instantons, euclidean D-strings, euclidean D2-branes (possibly), euclidean D3-branes and perhaps the most interesting ones are lorentzian D3-branes and space-filling D5-branes.

[3] A word of caution: we have set $\theta = 0$ because we are working in the universal cover of the group, so that $\theta_0 \in \mathbb{R}$. If instead we work in a quotient where $\theta_0 \in \mathbb{R}/2\pi n\mathbb{Z}$, then the condition is that $\theta \in \pi n\mathbb{Z}$, so that $e^{i\theta} = (-1)^n$, whence for odd n we would have to modify the discussion below.

JHEP06(2003)025

4.4 Geometry of lorentzian Lie branes

We will focus on the Lie branes with lorentzian signature and investigate their geometry. The space-filling D5-branes are of course isometric to the spacetime itself; hence we will concentrate on the D3-branes. The geometry of the D3-branes can be worked out from the formulae for the metric and the embedding of the twisted conjugacy classes, once we find a convenient parametrisation. We will show that the D-submanifolds wrapped by the D3-branes are isometric to the Nappi-Witten spacetime [5].

This is not unexpected, based on the analogous analysis of the Lie branes of the Nappi-Witten spacetime itself [11]. Indeed the twisted conjugacy classes of NW_4, whose metrics are given in [11, equations (22) and (23)], are themselves three-dimensional symmetric plane waves. Indeed they are isometric to the unique three-dimensional symmetric plane wave, as can be seen by changing variables in those equations to bring both metrics to the form

$$ds^2 = dx^2 - x^2 (dx^-)^2 + 2dx^+ dx^- \,.$$

We can also argue purely geometrically that D-submanifold isometric to NW_4 exists in NW_6. Indeed, as shown in [4], the Nappi-Witten metric arises as a Penrose limit of $AdS_2 \times S^2$ (with equal radii of curvature) along any null geodesic having a nonzero velocity component tangent to the sphere.[4] On the other hand, we know that $AdS_3 \times S^3$ has a family of D-branes with the geometry of $AdS_2 \times S^2$. We would like to find a situation in which the Penrose limit will simultaneously induce the Penrose limits of the brane and of the ambient spacetime. In other words, we would like to encounter a situation where the following diagram commutes

$$\begin{tikzcd} AdS_3 \times S^3 \arrow[r, "\text{Penrose limit}"] & \mathrm{NW}_6 \\ AdS_2 \times S^2 \arrow[r, "\text{Penrose limit}"] \arrow[u] & \mathrm{NW}_4 \arrow[u] \end{tikzcd} \tag{4.7}$$

where the vertical arrows are isometric embeddings given by the corresponding twisted conjugacy classes.

To achieve this, it is enough to ensure the existence a null geodesic in $AdS_3 \times S^3$ (with nonzero velocity component tangent to the sphere) which starts from a point in one of these $AdS_2 \times S^2$ and whose initial velocity is tangent to the $AdS_2 \times S^2$ *and* which stays on $AdS_2 \times S^2$ for all time. Then the Penrose limit of $AdS_3 \times S^3$ along this null geodesic, which gives rise to NW_6, induces simultaneously the Penrose limit of $AdS_2 \times S^2$, which is known to give rise to the Nappi-Witten spacetime NW_4. The above conditions on the null geodesic will be met if $AdS_2 \times S^2 \subset AdS_3 \times S^3$ is a *totally geodesic* submanifold; although this may be too strong, since we only require this property for a particular null geodesic. Let us recall that a submanifold is totally geodesic if and only if its second fundamental

[4] This is not the only way this metric arises as a Penrose limit: in [19] it arises as the Penrose limit of the near-horizon geometry of an NS5-brane.

JHEP06(2003)025

form vanishes. In principle one could compute the second fundamental forms of these (twisted) conjugacy classes and see that there is a totally geodesic $AdS_2 \times S^2$ among the twisted D-branes of $AdS_3 \times S^3$; but in fact we will arrive at this result by more geometric means.

To this end we employ the useful fact that a hypersurface which is the fixed point set of a "reflection" is automatically totally geodesic. Let us elaborate on this. Suppose that (M, g) is a (pseudo-)riemannian manifold and that $\tau : M \to M$ is an isometry having as fixed point set a hypersurface $N \subset M$; that is, a submanifold of codimension one. We will assume that g restricts non-degenerately on N, so that for every $p \in N$, $T_pM = T_pN \oplus T_pN^\perp$, and that the derivative map $\tau_* : T_pM \to T_pM$ is a reflection in T_pN. In other words, we have that

$$\tau_*|_{T_pN} = \mathrm{id} \qquad \text{and} \qquad \tau_*|_{T_pN^\perp} = -\,\mathrm{id}\ .$$

(This defines what we meant by "reflection" above.) The second fundamental form of $N \subset M$ defines for every $p \in N$ a symmetric bilinear map $\mathrm{II}_p : T_pN \times T_pN \to T_pN^\perp$. The action of τ_* is such that II_p changes sign, but on the other hand τ is an isometry which leaves N pointwise fixed, hence II_p is invariant. The only way these two statements can be reconciled is if II_p vanishes identically.

We will now exhibit a totally geodesic $\mathrm{AdS}_2 \times S^2$ in $\mathrm{AdS}_3 \times S^3$ by exhibiting the AdS_2 and the S^2 as fixed point sets of a reflection on AdS_3 and S^3, respectively. Let us first consider the $S^2 \subset S^3$ consisting of an equatorial sphere; in other words, if S^3 is defined as the quadric in $\mathbb{E}^4$ given by

$$x^2 + y^2 + z^2 + w^2 = R^2\ ,$$

then consider the reflection $x \mapsto -x$ and leaving y, z, w invariant. This is an isometry of $\mathbb{E}^4$ which preserves and hence induces an isometry in S^3. This isometry fixes the intersection of S^3 with the hyperplane $x = 0$, which is the equatorial two-sphere given by $x = 0$ and $y^2 + z^2 + w^2 = R^2$, which is then totally geodesic in S^3.

Similarly consider the following $\mathrm{AdS}_2 \subset \mathrm{AdS}_3$. If we think of AdS_3 as the quadric

$$x^2 + y^2 = R^2 + u^2 + v^2 \qquad \text{in } \mathbb{E}^{2,2}\ ,$$

then the family of twisted conjugacy classes isometric to AdS_2 are those corresponding to $u =$ constant. Consider the "time reversal" isometry of $\mathbb{E}^{2,2}$ given by $u \mapsto -u$ and leaving x, y, v invariant. This isometry preserves and hence induces an isometry of AdS_3 which fixes the intersection of AdS_3 with the hyperplane $u = 0$. The corresponding AdS_2 is therefore totally geodesic. Notice that both the totally geodesic S^2 and AdS_2 inherit their radius of curvature from the ambient S^3 and AdS_3, respectively; whence if the radius of curvature of the ambient spaces are equal, so are the ones of the AdS_2 and S^2. In summary we have exhibited a totally geodesic $AdS_2 \times S^2$ with equal radii of curvature inside $AdS_3 \times S^3$, and hence by the commutativity of the above diagram, the Penrose limit will give rise to a NW_4 submanifold of NW_6.

We can exhibit this submanifold explicitly as a twisted conjugacy class of the Nappi-Witten group $\mathcal{N}$, as follows. Consider the automorphism r of $\mathcal{N}$ given by

$$g(z, \theta, t) \mapsto g(\bar{z}, -\theta, -t)\ ;$$

JHEP06(2003)025

in other words, this is the element $(X, -1)$ in the outer automorphism group, corresponding to $S = \mathbb{1}$. Let $g_0 = g(z_0, \theta_0, t_0)$ be an arbitrary element and consider its orbit under the twisted adjoint action $r(g) g_0 g^{-1}$, whose elements are given by equation (4.4) with $S = \mathbb{1}$. Without loss of generality we parametrise the twisted conjugacy class as

$$g\left(e^{-i\phi/2} e^{i\theta_0/2} z_0 + \bar{z} - e^{-i\phi} z, \phi, s\right) ,$$

where $\phi, s \in \mathbb{R}$. As $z \in \mathbb{C}^2$ varies, the points

$$e^{-i\phi/2} e^{i\theta_0/2} z_0 + \bar{z} - e^{-i\phi} z$$

define an affine (real two-dimensional) plane through $e^{-i\phi/2} e^{i\theta_0/2} z_0$ in $\mathbb{R}^4$ which, after a moment's thought, can be seen to be parametrised by

$$e^{-i\phi/2} \left(e^{i\theta_0/2} z_0 - 2iy\right) \qquad \text{for } y \in \mathbb{R}^2 \subset \mathbb{C}^2 .$$

In summary, the twisted conjugacy class consists of elements

$$g\left(e^{-i\phi/2} \left(e^{i\theta_0/2} z_0 - 2iy\right), \phi, s\right) ,$$

where $(y, \phi, t) \in \mathbb{R}^4$. Write $e^{i\theta_0/2} z_0 = x_0 + i y_0$, with $x_0, y_0 \in \mathbb{R}^2$. Define the following coordinates

$$x = y_0 - 2y , \qquad x^- = \frac{1}{2}\phi \qquad \text{and} \qquad x^+ = -2s + \frac{1}{4}|x_0|^2 \phi ,$$

where here and below $|\;|^2$ indicates the euclidean norm of a two-vector. Relative to these coordinates the metric induced from (2.11) takes the form

$$ds^2 = |dx|^2 - |x|^2 (dx^-)^2 + 2 dx^+ dx^- , \tag{4.8}$$

which we recognise as the metric of the Nappi-Witten spacetime. In summary, we have exhibited a foliation of the Penrose limit of $\mathrm{AdS}_3 \times S^3$ consisting of Nappi-Witten spacetimes. Moreover (at least some of) these "braneworlds" are themselves the Penrose limits of $AdS_2 \times S^2$ braneworlds in $AdS_3 \times S^3$.

How about the other lorentzian D3-branes? In fact, it is not hard to show that they are also isometric to Nappi-Witten spacetimes. To see this let $S = \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$, where $b = i\gamma$ is pure imaginary and $a = \alpha + i\beta$, where $\alpha^2 + \beta^2 + \gamma^2 = 1$. We will assume that $\alpha < 1$, for otherwise $S = \mathbb{1}$ and this was the Lie brane we just described. The corresponding twisted conjugacy class consists of points

$$g\left(e^{-i\phi/2} e^{i\theta_0/2} z_0 + S\bar{z} - e^{-i\phi} z, \phi, s\right) ,$$

with $\phi, s \in \mathbb{R}$. It is convenient to define $w = e^{-i\phi/2} z$ and $w_0 = e^{i\theta_0/2} z_0$, so that the conjugacy class now consists of the points

$$g\left(e^{-i\phi/2} \left(w_0 + S\bar{w} - w\right), \phi, s\right) .$$

JHEP06(2003)025

Our assumption that $\alpha \neq 1$ implies that we can take $w \in \mathbb{R}^2 \subset \mathbb{C}^2$ and this still parametrises the conjugacy class. (For $\alpha = 1$ we would (and did) take $iw \in \mathbb{R}^2$.) The induced metric on this submanifold is given by

$$ds^2 = |Sdw - dw|^2 - |w_0 + Sw - w|^2 d\phi^2 - 2d\phi ds\,,$$

which can be rewritten as

$$ds^2 = 2(1-\alpha)|dw|^2 - \left(2\,(1-\alpha)\,|w-c|^2 + \mu\right) d\phi^2 - 2d\phi ds\,,$$

for some $c \in \mathbb{R}^2$ and $\mu \in \mathbb{R}$ whose explicit expressions are of no relevance. Now simply define the new coordinates

$$x = \sqrt{2(1-\alpha)}(w-c)\,, \qquad x^- = \frac{1}{2}\phi \qquad \text{and} \qquad x^+ = -2s - \mu\phi\,,$$

relative to which the metric adopts the standard form (4.8) for the Nappi-Witten spacetime.

5. Summary and discussion

In summary we have determined the Lie branes of the symmetric plane wave NW_6 which arises as the Penrose limit of $AdS_3 \times S^3$ along a generic null geodesic. Among the lorentzian Lie branes we have found D3-branes which are isometric to Nappi-Witten spacetimes NW_4. Indeed, NW_6 can be foliated by NW_4 in a variety of ways. We have argued that at least one of these D3-branes arises as the Penrose limit of a totally geodesic $AdS_2 \times S^2$ Lie brane of $AdS_3 \times S^3$. We also exhibited the whole NW_6 spacetime as a space-filling D5-brane, but since there are no space-filling Lie branes in $AdS_3 \times S^3$, its origin is more mysterious; although as shown in [16], when warped into a fivebrane solution of type IIB supergravity it can be understood as the Penrose limit of a (non Lie) fivebrane solution with worldvolume $AdS_3 \times S^3$.

One can ask about the fate of the other Lie branes in $AdS_3 \times S^3$ under the Penrose limit. If the null geodesic along which the limit is taken does not lie along the Lie brane, there is not much that can be said. Some of the D-strings in $AdS_3 \times S^3$, e.g., those with AdS_2 geometry, behave nicely under the Penrose limit of $AdS_3 \times S^3$ along a null geodesic with zero velocity component along S^3, which yields flat space. These D-strings become D-strings with flat worldsheets in Minkowski spacetime. Notice that if we think of Minkowski spacetime as an abelian Lie group, then these Lie branes are precisely the twisted conjugacy classes which, when written additively, are seen to be precisely the affine planes consisting of the points

$$x_0 + r(x) - x\,,$$

where r is now a Lorentz transformation. Similarly, it is tempting to conjecture that lorentzian twisted conjugacy classes of lorentzian Lie groups which are also symmetric plane waves are themselves symmetric plane waves.

JHEP06(2003)025

Acknowledgments

The research of S.S. was funded by PPARC under grant SPG/00613, whereas that of J.M.F. is supported in part by EPSRC under grant GR/R62694/01. J.M.F. is a member of EDGE, Research Training Network HPRN-CT-2000-00101, supported by The European Human Potential Programme. J.M.F. would like to thank the Rutgers NHETC for hospitality during the final stages of the writing of this paper.

References

[1] R. Penrose, *Any space-time has a plane wave as a limit*, in *Differential geometry and relativity*, pp. 271–275. Reidel, Dordrecht 1976.

[2] R. Güven, *Plane wave limits and T-duality*, *Phys. Lett.* **B 482** (2000) 255 [hep-th/0005061].

[3] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and PP waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[4] M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos, *Penrose limits, supergravity and brane dynamics*, *Class. and Quant. Grav.* **19** (2002) 4753 [hep-th/0202111].

[5] C.R. Nappi and E. Witten, *A WZW model based on a nonsemisimple group*, *Phys. Rev. Lett.* **71** (1993) 3751 [hep-th/9310112].

[6] P. Meessen, *A small note on PP-wave vacua in 6 and 5 dimensions*, *Phys. Rev.* **D 65** (2002) 087501 [hep-th/0111031].

[7] E. Inönü and E. Wigner, *On the contraction of groups and their representations*, *Proc. Nat. Acad. Sci. USA* **39** (1956) 510.

[8] A.Y. Alekseev and V. Schomerus, *D-branes in the WZW model*, *Phys. Rev.* **D 60** (1999) 061901 [hep-th/9812193].

[9] G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, *The geometry of WZW branes*, *J. Geom. Phys.* **34** (2000) 162 [hep-th/9909030].

[10] S. Stanciu, *D-branes in group manifolds*, *J. High Energy Phys.* **01** (2000) 025 [hep-th/9909163].

[11] J.M. Figueroa-O'Farrill and S. Stanciu, *More D-branes in the Nappi-Witten background*, *J. High Energy Phys.* **01** (2000) 024 [hep-th/9909164].

[12] S. Stanciu, *D-branes in an AdS_3 background*, *J. High Energy Phys.* **09** (1999) 028 [hep-th/9901122].

[13] C. Bachas and M. Petropoulos, *Anti-de Sitter D-branes*, *J. High Energy Phys.* **02** (2001) 025 [hep-th/0012234].

[14] A. Biswas, A. Kumar and K.L. Panigrahi, *p-p' branes in PP-wave background*, *Phys. Rev.* **D 66** (2002) 126002 [hep-th/0208042].

[15] R.R. Nayak, *D-branes at angle in PP-wave background*, *Phys. Rev.* **D 67** (2003) 086006 [hep-th/0210230].

[16] A. Kumar, R.R. Nayak and Sanjay, *D-brane solutions in PP-wave background*, *Phys. Lett.* **B 541** (2002) 183 [hep-th/0204025].

JHEP06(2003)025

[17] H. Singh, *M5-branes with 3/8 supersymmetry in PP-wave background*, hep-th/0205020.

[18] M. Alishahiha and A. Kumar, *D-brane solutions from new isometries of PP-waves*, *Phys. Lett.* **B 542** (2002) 130 [hep-th/0205134].

[19] J. Gomis and H. Ooguri, *Penrose limit of $N = 1$ gauge theories*, *Nucl. Phys.* **B 635** (2002) 106 [hep-th/0202157].

[20] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, *Class. and Quant. Grav.* **19** (2002) L87 [hep-th/0201081].

[21] D.I. Olive, E. Rabinovici and A. Schwimmer, *A class of string backgrounds as a semiclassical limit of WZW models*, *Phys. Lett.* **B 321** (1994) 361 [hep-th/9311081].

[22] A. Medina and P. Revoy, *Algèbres de Lie et produit scalaire invariant*, *Ann. Scient. Éc. Norm. Sup.* **18** (1985) 553.

[23] J.M. Figueroa-O'Farrill and S. Stanciu, *Nonsemisimple Sugawara constructions*, *Phys. Lett.* **B 327** (1994) 40 [hep-th/9402035].

[24] M. Cahen and N. Wallach, *Lorentzian symmetric spaces*, *Bull. Am. Math. Soc.* **76** (1970) 585.

[25] J.M. Figueroa-O'Farrill and S. Stanciu, *On the structure of symmetric selfdual Lie algebras*, *J. Math. Phys.* **37** (1996) 4121 [hep-th/9506152].

[26] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, *J. High Energy Phys.* **01** (2002) 047 [hep-th/0110242].

[27] J. Kowalski-Glikman, *Vacuum states in supersymmetric Kaluza-Klein theory*, *Phys. Lett.* **B 134** (1984) 194.

[28] J. Figueroa-O'Farrill and G. Papadopoulos, *Plücker-type relations for orthogonal planes*, math.AG/0211170.

JHEP06(2003)025